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Log-Concavity of Whitney Numbers of Dowling Lattices

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We prove that the generating polynomial of Whitney numbers of the second kind of Dowling lattices has only real zeros. © 1999 Academic Press

1. INTRODUCTION

Let G be a finite group of order $m \geq 1$, let $Q_n(G)$ be the Dowling lattice associated to G, and let $w_m(n,k)$, $0 \leq k \leq n$, be the Whitney numbers of the second kind of $Q_n(G)$.

It is a long standing conjecture that the Whitney numbers of the second kind of any finite geometric lattice are a log-concave sequence ([8], Conjecture 3, p. 508). Recall that the $w_m(n,k)$ satisfy the following recursion formula, (see [4]),

$$w_m(n,k) = w_m(n-1,k-1) + (mk+1)w_m(n-1,k), \tag{1}$$

with the boundary conditions,

$$w_m(n,n) = w_m(n,0) = 1, \quad \text{for } n \geq 0,$$
$$w_m(n,k) = 0, \quad \text{if } k > n \quad \text{or} \quad n < 0.$$

Using (1), Stonesifer [9] proved that

$$\begin{aligned} w_m^2(n,k) &\geq \frac{n-k+1}{n-k}\,\frac{k+1}{k} w_m(n,k-1)w_m(n,k+1), \qquad k=1,\dots,n-1,\\ &> w_m(n,k-1)w_m(m,k+1), \qquad k=1,\dots,n-1. \end{aligned} \tag{2}$$

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That is, the sequence $(w_m(n,k))_{k=0}^n$ is strictly log-concave (SLC); i.e., inequalities in (2) are strict, recall that in this case the sequence has at most two consecutive maximas see [2]. Damiani, D'Antona, and Regonati [3] proved (differently) that this sequence is log-concave rediscovering the result in [9]. The following result is known and goes back to Newton, for a proof see [5, Sections 2.22 and 4.3].

THEOREM 1. *Let $(a_k)_{k=0}^n$ be a sequence of real numbers; assume that the polynomial $P(x) = \sum_{k=0}^n a_k x^k$ has only real roots then*

$$a_k^2 \geq \frac{n-k+1}{n-k}\,\frac{k+1}{k}\,a_{k+1}a_{k-1}, \qquad k = 1, \dots, n-1.$$

Obviously, if $(a_k)_{k=0}^n$ is positive then it is S. L. C.

In this article, we prove that the polynomial $P_n(x) = \sum_{k=0}^n w_m(n,k)x^k$ has only real negative zeros, i.e., $(w_m(n,k))_{k\geq 0}$, is a PF sequence, which is more general than the results in [3] and [9], because these results assert that the previous sequence is PF_2. For more information on PF sequences see [1]. The main result is

THEOREM 2. *Let $P_n(x) = \sum_{k=0}^n w_m(n,k)x^k$ be the polynomial associated to the sequence $(w_m(n,k))_{k=0}^n$. Then all the zeros of $P_n(x)$ are real and negative.*

COROLLARY. *The sequence $(w_m(n,k))_{k=0}^n$ satisfies the inequalities,*

$$w_m^2(n,k) \geq \frac{n-k+1}{n-k}\,\frac{k+1}{k}\,w_m(n,k-1)w_m(n,k+1), \qquad k=1,\dots,n-1.$$

Proof. By the previous theorem the polynomial $P_n(x) = \sum_{k=0}^n w_m(n,k)x^k$ has only real zeros, and by Theorem 1, we obtain the desired result.

2. PROOF OF THEOREM 2

We proceed by induction on n; for small n we have by (1),

$$w_m(1,0) = 1, \qquad w_m(1,1) = 1.$$

$$w_m(2,0) = 1, \qquad w_m(2,1) = m+2, \qquad w_m(2,2) = 1.$$

$$w_m(3,0) = 1, \qquad w_m(3,1) = m^2 + 3m + 3,$$

$$w_m(3,2) = 3m+3, \qquad w_m(3,3) = 1.$$

So, $P_1(x) = 1 + x$, $P_2(x) = 1 + (m+2)x + x^2$, and the result is trivially true for $n = 1, 2$. Assume now that the result holds for $n-1$, and set $P_n(x) = \sum_{k=0}^{n} w_m(n,k)x^k$. Using identity (1), we obtain

$$P_n(x) = (x+1)P_{n-1}(x) + mxP'_{n-1}(x).$$

Define the function $H_{n,m}$ as follows,

$$H_{n,m}(x) = xe^x P_{n-1}^m(x).$$

By the induction hypothesis the function $H_{n,m}(x)$ has $1 + m(n-1)$ finite real zeros, and an infinite one at $-\infty$, so the derivative of $H_{n,m}$, which is

$$H'_{n,m}(x) = e^x P_{n-1}^{m-1}(x) P_n(x) \tag{3}$$

has $1 + m(n-1)$ finite real zeros. Indeed, by Rolle's theorem between two finite zeros of $H_{n,m}$, there is one of $H'_{n,m}(x)$, this gives $m(n-1)$, and between $-\infty$ and the least zero of $H_{n,m}$ there is another one, this is what we claimed. Now if $N(p)$ denote the number of real zeros of $P_n(x)$, then by the induction hypothesis and (3) we have

$$mn - m + 1 = (m-1)(n-1) + N(P),$$

that is $N(P) = n$, and the proof is achieved.

Remark. If G is the trivial group $(m = 1)$, then $Q_n(G)$ is isomorphic to $\prod_{n+1}$, the lattice of partitions of an $(n+1)$-element set. In this case the Whitney numbers of the second kind are the Stirling numbers of the second kind. The fact that the polynomial associated to these numbers have only real zeros is known because Harper [6], to prove this fact, used induction and the auxiliary function $H_n(x) = xe^x P_n(x)$, which follows if we set $m = 1$ in $H_{n,m}(x)$.

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